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Contraction Mapping Principle

Recall from the previous lectures, we learnt about:

Definition 1
A map $T : (X, d) \to (X, d)$ is a *contraction* if there exists a constant $\gamma \in (0, 1)$ such that

$$d(Tx, Ty) \leq \gamma d(x, y)$$

for all $x, y \in X$.

Theorem 1 (Contraction mapping principle)
Every contraction in a complete metric space admits a unique fixed point.

In this tutorial, we prove the following corollary:

Corollary 1 (Source: Functional Analysis by S. Kesavan P.55)
Let (X, d) be a complete metric space and let $T : (X, d) \to (X, d)$ be a map such that for some positive integer n, the map $T^n = T \circ \cdots \circ T : (X, d) \to (X, d)$ is a contraction. Then T has a unique fixed point.

Proof of Corollary 1:
Denote the fixed point of T^n by x^*, then

$$Tx^* = TT^n x^* = T^{n+1} x^* = T^n T x^*$$

that means Tx^* is also a fixed point of T^n. However the contraction mapping principle tells us that the fixed point x^* is unique. This implies $Tx^* = x^*$.

Uniqueness follows from the fact that any fixed point x^* of T is also a fixed point of T^n, because

$$T^n x^* = T^{n-1} T x^* = T^{n-1} x^* = \cdots = Tx^* = x^*$$

Thus T has unique fixed point.

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Exercise 1

Show that every continuous function $f : [0,1] \to [0,1]$ has a fixed point.

Solution:

If $f(0) = 0$ or $f(1) = 1$, then we are done. Hence, we assume $f(0) > 0$ and $f(1) < 1$. Define $g : [0,1] \to \mathbb{R}$ by $g(x) = f(x) - x$. Then $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$.

Since f is continuous, then g is continuous. By the intermediate value theorem, there exists a $c \in [0,1]$ such that

$$g(c) = f(c) - c = 0 \implies f(c) = c$$

thus, f has a fixed point

$\blacksquare$

Exercise 2

(Source: Previous HW problem of MATH3060)
Fix $\alpha \in [0,1)$, for each $x_0 \in [0,1]$, consider the iteration sequence

$$x_n = \alpha x_{n-1}(1 - x_{n-1}), \forall n \in \mathbb{N}$$

(a) Show that $\{x_n\} \subseteq [0,1]$

(b) Show that $\lim_{n\to\infty} x_n = 0$

Solution:

We first show that $\{x_n\} \subseteq [0,1]$. Define $T : [0,1] \to \mathbb{R}$, by $Tx = \alpha x(1-x)$. Then the iteration sequence can be written as

$$x_n = Tx_{n-1}$$

Then it is equivalent to show that $T([0,1]) \subset [0,1]$.

Since T is smooth, we differentiate T and get $T'(x) = \alpha(1-2x)$ which implies that T attains its maximum at $x = \frac{1}{2}$, and the value is $T(\frac{1}{2}) = \frac{\alpha}{4} < 1$ since $\alpha < 1$. Moreover, it is obvious that $Tx \geq 0$ for all $x \in [0,1]$. Thus, $T([0,1]) \subset [0,1]$. It follows that $x_n = T^n x_0 \in [0,1]$ for all n.

First of all we know that $T(0) = 0$. Then we check that T is a contraction

$$|Tx - Ty| = |T'(c)||x-y| \leq M|x-y|$$

where $M := \max_{x\in[0,1]} |T'(x)|$, which can be calculated to be $\alpha < 1$. Hence for all $x, y \in [0,1]$, T is a contraction. Then the contraction mapping principle tell us that the fixed point $x = 0$ is unique.

Now suppose that $x_n \to L$ as $n \to \infty$, then

$$L = \lim_{n\to\infty} x_n = T(\lim_{n\to\infty} x_{n-1}) = T(L) \implies L = 0$$

by the contraction mapping principle.

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Exercise 3

(Source: Ordinary Differential Equations, Lecture Notes of MATH4051 at HKUST, by Prof Frederick Fong.)

Define $g(x) = \cos x - \frac{1}{3}\cos^3 x$ where $x \in [0, \frac{\pi}{3}]$. Consider the iteration sequence

$$\begin{cases} x_0 = 0 \\ x_n = g(x_{n-1}) \end{cases}, \forall n \geq 1$$

Show that $g(x)$ maps $[0, \frac{\pi}{3}]$ to $[0, \frac{\pi}{3}]$ and that it satisfies the contraction inequality. Hence, show that the iteration sequence x_n converges to a limit L which is the root of the equation $x = \cos x - \frac{1}{3}\cos^3 x$.

Solution:
Since $g(x)$ is smooth, we can consider its derivative:

$$\begin{aligned} g'(x) &= -\sin x - \cos^2 x(-\sin x) \\ &= -\sin x(1 - \cos^2 x) \\ &= -\sin^3 x \end{aligned}$$

then one can see that $g'(x) \leq 0$ for all $x \in [0, \frac{\pi}{3}]$, hence it is decreasing.

Note that $g(0) = 1 - \frac{1}{3} = \frac{2}{3} < \frac{\pi}{3}$, and that $g(\frac{\pi}{3}) = \frac{1}{2} - \frac{1}{24} = \frac{11}{24} > 0$. Together with g the fact that is decreasing, we see that $g([0, \frac{\pi}{3}]) \subset [0, \frac{\pi}{3}]$.

Then for all $x, y \in [0, \frac{\pi}{3}]$, by the mean value theorem, one can see that

$$|g(x) - g(y)| = |g'(c)||x - y| \leq M|x - y|$$

where $M := \max_{x \in [0, \frac{\pi}{3}]} |g'(x)|$. On $[0, \frac{\pi}{3}]$, we have $M = |\sin^3(\frac{\pi}{3})| = \frac{3\sqrt{3}}{8} < 1$

Now consider the iteration sequence $x_n = g(x_{n-1})$, we have

$$\begin{aligned} |x_{n+1} - x_n| &= |g(x_n) - g(x_{n-1})| \\ &\leq M|x_n - x_{n-1}| \\ &\leq M^2|x_{n-1} - x_{n-2}| \\ &\leq M^{n-1}|x_2 - x_1| \end{aligned}$$

Then we see that

$$\sum_{n=0}^{\infty} |x_{n+1} - x_n| \leq \sum_{n=0}^{\infty} M^{n-1}|x_2 - x_1|$$

converges, since the RHS is a geometric series and $M < 1$. Then the series

$$\sum_{n=0}^{\infty} (x_{n+1} - x_n)$$

converges absolutely. Hence $x_N := x_1 + \sum_{n=1}^{N-1}(x_{n+1} - x_n)$ converges as $N \to \infty$.

Suppose that $x_n \to L$ as $n \to \infty$, then

$$L = \lim_{n\to\infty} x_n = g(\lim_{n\to\infty} x_n) = g(L)$$

implies that L is a root of the equation $x = g(x)$.

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Possible Reference

Instead of giving you another exercise, the following books contain many examples in which you can take a look at them if you are interested:

- Real Analysis by Royden and Fitzpatrick
- Metric Spaces by Copson